Rocket Thermodynamics

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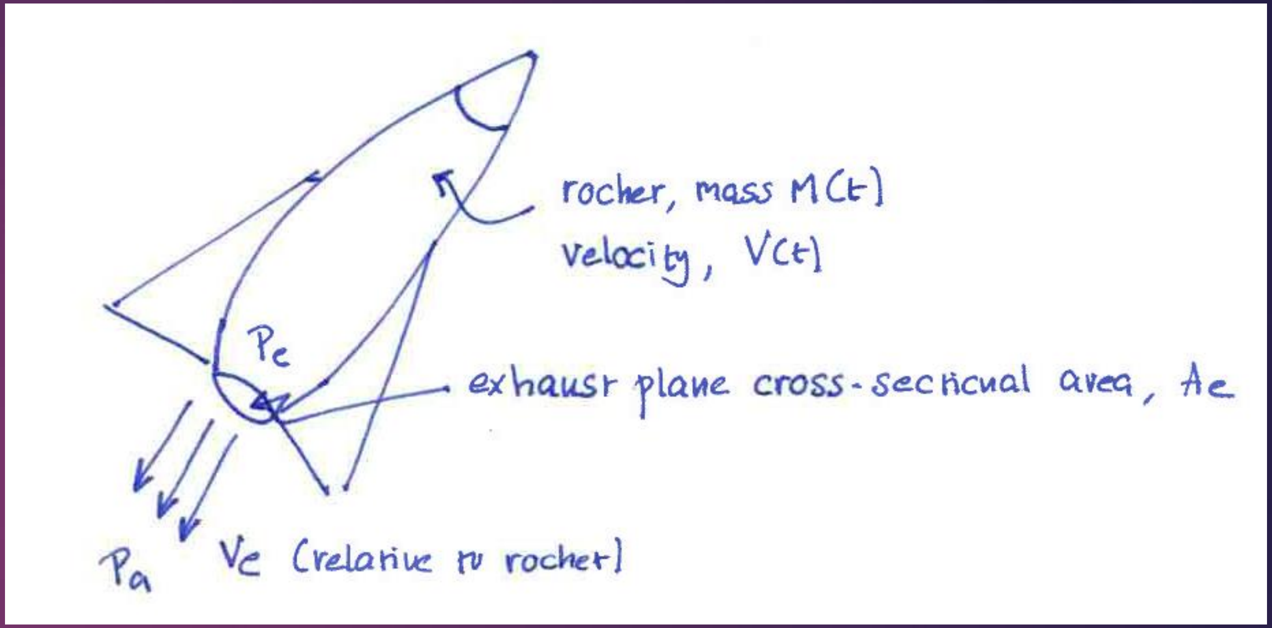
25TH APRIL 2017

Thermodynamics of Chemical Rockets



- $\Sigma Force = mass \times acceleration$
- $M\frac{dV}{dt} = \dot{m}V_e + A_e(P_e - P_a) + F_{ext}$
- Rocket Thrust = $\dot{m}V_e + A_e(P_e - P_a)$
- The rocket thrust comprises:
- Exhaust momentum thrust: $\dot{m}V_e$
- Exhaust plane pressure difference: $A_e\,(P_e - P_a)$
- Also from the Rocket equation: $\Delta V = V_e\, log_e\left(\frac{M_0}{M}\right)$
- Hence the need to predict, V_e and P_e to calculate rocket performance or do design

Continuity equation

- Continuity equation for 1 Dimensional flow
- $\dot{\mathrm{m}} = \rho_x u_x A_x = constant$
- Differenrtiate (product rule)
- $\frac{d\dot{\mathrm{m}}}{dx} = 0 = \frac{d(\rho u A)}{dx} = uA\frac{d\rho}{dx} + u\rho\frac{dA}{dx} + \rho A\frac{du}{dx}$
- Divide through by $(\rho u A)$
- $\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0$ (1)

Momentum and Sonic velocity

- Momemtum equation
  - Eulers equation $\rho\left\{u\frac{du}{dx}+v\frac{dv}{dy}\right\}=-\frac{dP}{dx}+viscous\ terms$
  - $v=0\ as\ 1D\quad viscous\ terms=zero\ as\ isentropic$
  - $\frac{dP}{\rho}+udu=0$ (2)
- And sonic velocity $a^2=\frac{dP}{d\rho}$
- $dP=a^2d\rho$ (3)
- Divide (2) by u^2
- $\frac{dP}{\rho u^2}=-\frac{du}{u}$ substitute into (1)
- (this relationship uses three independent fluid flow relations: continuity, momentum & sonic velocity)

Flow behavior

- Into (1)
- $\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0$ (1)
- $\frac{d\rho}{\rho} + \frac{dA}{A} - \frac{dP}{\rho u^2} = 0$
- $\frac{dA}{A} = \frac{dP}{\rho u^2} - \frac{d\rho}{\rho} \quad = \frac{dP}{\rho u^2}\left\{1 - \frac{u^2}{dP/d\rho}\right\}$
- From (3) with mach number M = u/a
- $\frac{dA}{A} = \frac{dP}{\rho u^2}\{1 - M^2\} \quad = -\frac{du}{u}(1 - M^2)$
- Rearrange
- $\frac{du}{u} = \frac{dA}{A}\frac{1}{(M^2-1)} = -\frac{dP}{\rho u^2}$
- Inspection of this equation reveals some interesting and physically important characteristics of compressible fluid flow

Nozzle flow behaviour

- $\frac{du}{u} = \frac{dA}{A}\frac{1}{(M^2-1)} = -\frac{dP}{\rho u^2}$
- Subsonic, M < 1
  - $dA > 0$, area increases, $du < 0$ therefore u decreases, P increases
  - $dA < 0$, area decreases, $du > 0$ therefore u increases, P decreases
- Supersonic, $M > 1$
  - $dA > 0$, area increases, $du > 0$ therefore u increases, P decreases
  - $dA < 0$, area decreases, $du < 0$ therefore u decreases, P increases
  - Also: when $u = a$, $M = 1$, $dA = 0$ and A is a minimum

Nozzle flow behaviour

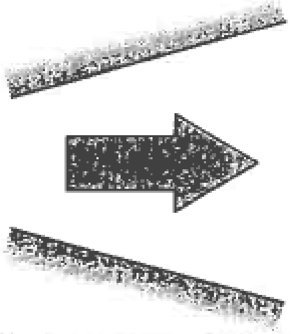
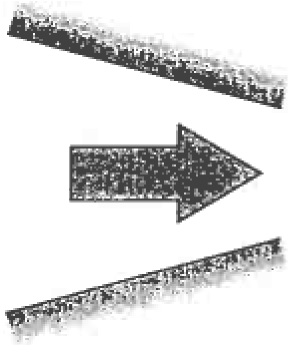
Duct geometry		*Subsonic* Ma < 1	*Supersonic* Ma > 1
	$dA > 0$	$dU < 0$ $dp > 0$ Subsonic diffuser	$dU > 0$ $dp < 0$ Supersonic nozzle
	$dA < 0$	$dU > 0$ $dp < 0$ Subsonic nozzle	$dU < 0$ $dp > 0$ Supersonic diffuser

Fig. 9.5 Effect of Mach number on property changes with area change in duct flow.

Why do we need a convergent-divergent nozzle?

- Note the opposite effects that occur in supersonic ($M > 1$) flow, compared with sonic flow due to the (M^2 -1) term
- What is it about $M = 1$? , at $M = 1$ du is infinite, which is not possible. But du can be made finite only if $dA = 0$. ($dA= 0$)
- Hence to make a subsonic flow supersonic, we use a convergent-divergent nozzle.

Thermodynamics revision

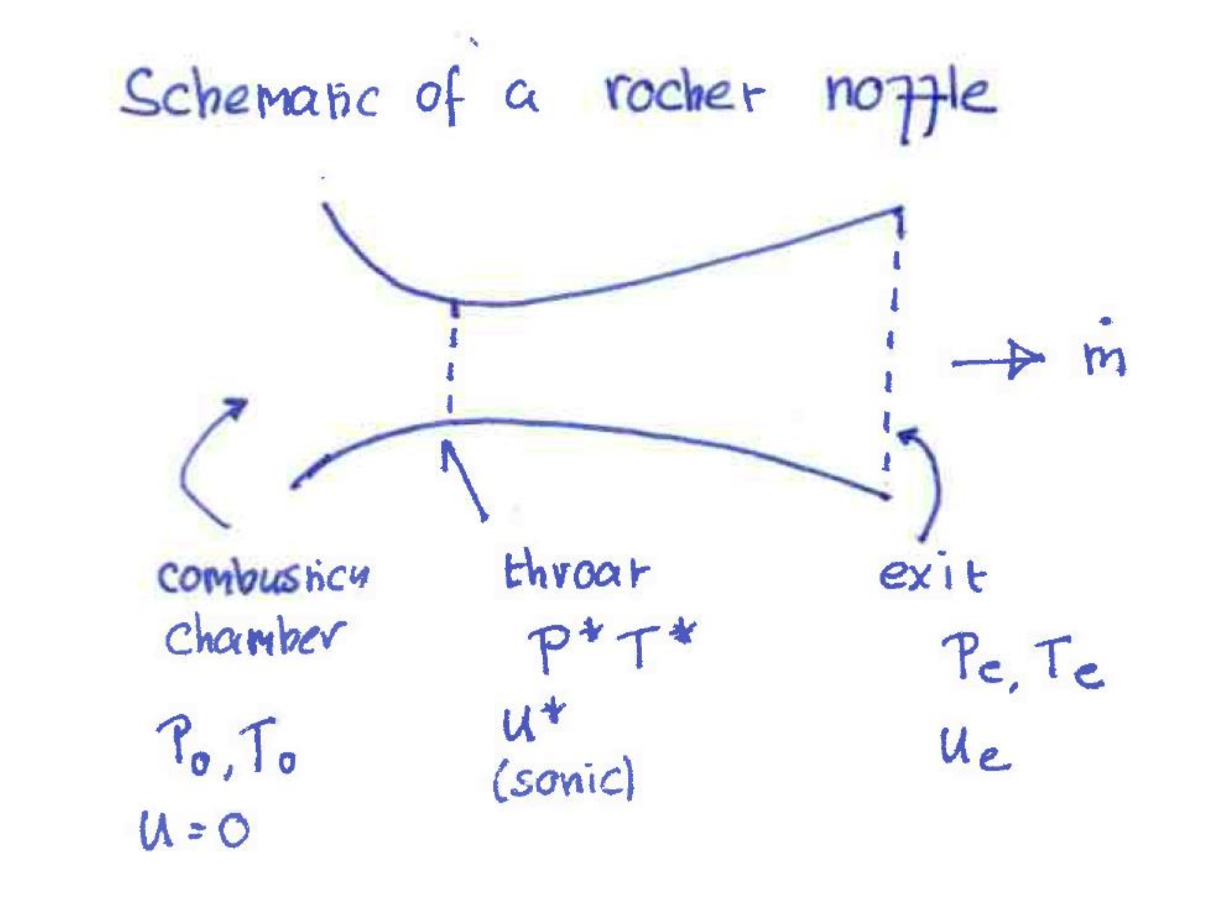
- $\rho = \frac{m}{V};$ $PV = mRT$ or $\frac{P}{\rho} = RT$
- $R = R_o/MW$
- $\frac{PV}{T} = constant,$ $PV^n = constant,$
- $\frac{P}{\rho^n} = constant$ used for a flow process
- First Law of thermodynamics: $Q - W = \Delta E$
- For an isentropic process $n = \gamma = C_p / C_v;$ $C_p - C_v = R$

Thermodynamics revision

- *Conservation of mass* $\dot{m} = \rho u A$
- *h (enthalpy)* $= C_p T$
- *sonic velocity* $a = (\gamma RT)^{1/2}$; *Mach number* $M = u/a$
- *Total and static quantities*
- $h_o = h_{exit} = constant$
- $T_o = T + \frac{u^2}{2C_p} = constant$
- $\frac{P}{\rho^{\gamma}} = constant$
- $\left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{P_o}{P}\right)$

Constant enthalpy; zero work, zero heat transfer

- Chemical rockets work by burning the fuel and oxidizer in a combustion chamber. This generates a high pressure, high temperature gas that is expanded through a convergent-divergent nozzle,producing a high (supersonic) exit velocity.
- Applying the first law of thermodynmamics to this flow: Q – W = ΔE
- Q = 0 , W = 0
- $h_0 = h_{exit} = constant\ enthalpy$
- $T_0 = T + \frac{u^2}{2C_p}$
- $T_0 = total\ or\ stagnation\ temperature$
- $T = static\ temperature$

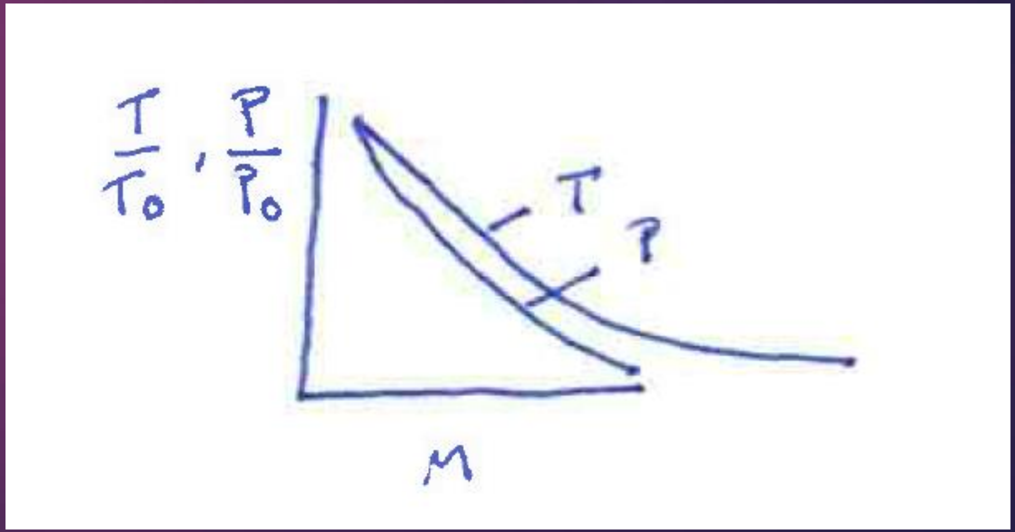


Total Temperature also the Combustion Chamber temperature

- $\frac{P}{\rho^{\gamma}} = constant$
- $\rho = \frac{P}{RT}$, R = R_o/MW R_0 = 8314 J/kg K , MW is the molecular weight of the fuel
- $\left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{P_o}{P}\right)$
- Also since *Mach number* $M = u/a$
- *And* $a = (\gamma RT)^{1/2}$
- $T_0 = T + \frac{u^2}{2C_p}$; *hence* $\frac{T_0}{T} = 1 + \frac{u^2}{2C_p\,T} = 1 + \frac{M^2 a^2}{2C_p\,T} = 1 + \frac{M^2 \gamma RT}{2C_p\,T}$

Total pressure also the Combustion Chamber pressure

- $\frac{T_0}{T} = 1 + \frac{M^2 \gamma R T}{2 C_p T}$
- $C_P = C_V + R$
- $\gamma = \frac{C_P}{C_V}$; hence $C_P = \frac{\gamma R}{\gamma - 1}$
- $\frac{T_0}{T} = 1 + \frac{M^2 \gamma R T (\gamma - 1)}{2 \gamma R T} = 1 + \frac{M^2 (\gamma - 1)}{2}$
- Using $\left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{P_o}{P}\right)$
- $\left(\frac{P_o}{P}\right) = \left(1 + \frac{M^2 (\gamma - 1)}{2}\right)^{\frac{\gamma}{\gamma - 1}}$



Choked mass flow rate

- For supersonic flow, sonic velocity is achieved at the nozzle throat, the velocity here is then
- $u^* = (\gamma R T^*)^{\frac{1}{2}}$ for the choked condition M = 1
- The velocity increases after the throat, the flow rate is:
- $\dot{m} = \rho u A$ *using the condition at the throat*
- $\dot{m} = \frac{P^*}{RT^*}(\gamma R T^*)^{1/2} A^*$
- $\left(\frac{P_o}{P^*}\right) = \left(1 + \frac{(\gamma - 1)}{2}\right)^{\frac{\gamma}{\gamma - 1}}$; M = 1
- $\left(\frac{T_o}{T^*}\right) = \left(1 + \frac{(\gamma - 1)}{2}\right)$; M = 1

Throat conditions and mass flow

- $P^* = \dfrac{P_0}{\left(1+\frac{(\gamma-1)}{2}\right)^{\frac{\gamma}{\gamma-1}}}$
- $T^* = \dfrac{T_0}{\left(1+\frac{(\gamma-1)}{2}\right)}$
- $\dot{m} = \dfrac{P^*}{RT^*}(\gamma RT^*)^{1/2}A^*$
- $\dot{m} = \dfrac{P^*}{R(T^*)^{1/2}}(\gamma R)^{1/2}A^*$
- $\dot{m} = P^*\left(\dfrac{\gamma}{RT^*}\right)^{1/2}A^*$
- And using the preceding relationships to express this in terms of total pressure and temperature (ie combustion chamber conditions)
- $\dot{m} = \dfrac{P_0}{\left(1+\frac{(\gamma-1)}{2}\right)^{\frac{\gamma}{\gamma-1}}}\left(\dfrac{\gamma MW}{R_0}\right)^{1/2}\left(\dfrac{1+\frac{(\gamma-1)}{2}}{T_0}\right)^{1/2}A^*$

Re-arranging Mass Flow equation

- $1+\frac{\gamma-1}{2} = \frac{2+\gamma-1}{2} = \frac{\gamma+1}{2} = \frac{1}{\emptyset}$
- $\dot{m} = \frac{P_0}{(T_0)^{1/2}}(\emptyset)^{\gamma/\gamma-1}\left(\frac{\gamma MW}{R_0}\right)^{1/2}(\emptyset)^{-1/2}A^*$
- $\dot{m} = (\emptyset)^{\gamma/\gamma-1-1/2}\frac{P_0}{(T_0)^{1/2}}\left(\frac{\gamma MW}{R_0}\right)^{1/2}A^*$
- $\frac{\gamma}{\gamma-1} - \frac{1}{2} = \frac{2\gamma-\gamma+1}{2(\gamma-1)} = \frac{\gamma+1}{2(\gamma-1)}$
- So
- $\dot{m} = \left\{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} P_0 A^* \left(\frac{\gamma MW}{R_0 T_0}\right)^{1/2}\right\}$

Example 7.1

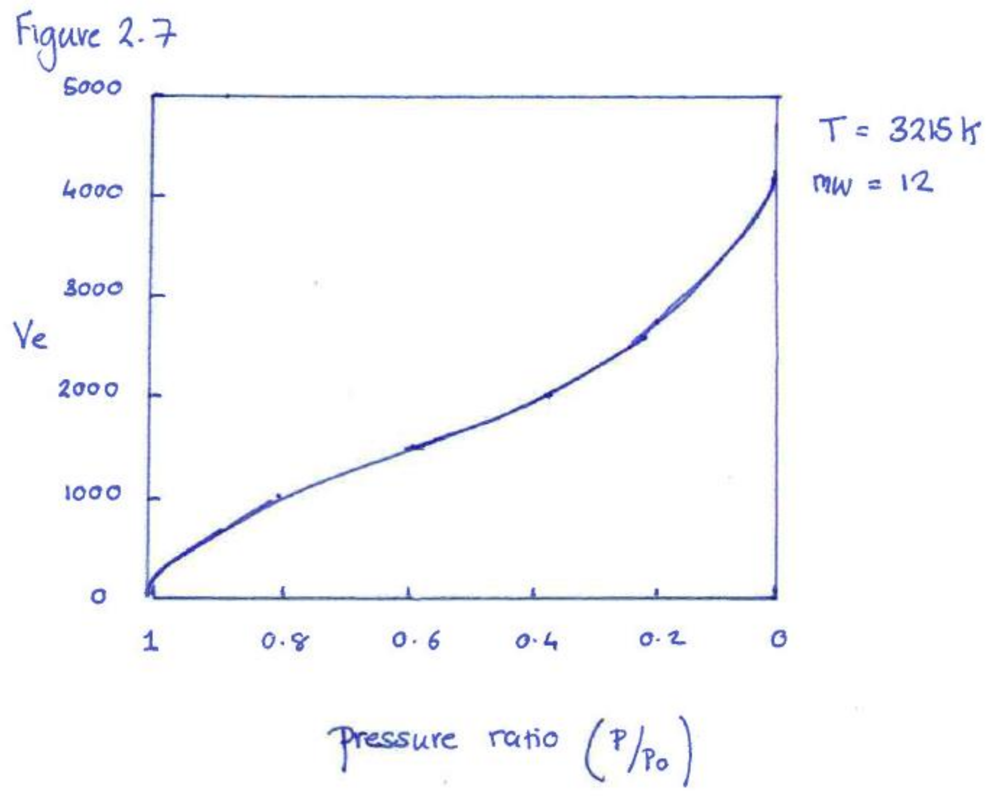
Characteristic velocity

- The Characteristic Velocity c^* is given by
- $c^* = \frac{P_0 A^*}{\dot{m}} = \frac{\left(\frac{R_0 T_0}{MW}\right)^{1/2}}{\gamma^{1/2}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$ This tells us about performance of the propellant, it measures the efficiency of conversion of thermal to kinetic energy
- Returning now to the problem of calculating the exit velocity.
- Between the combustion chamber and exit
- $C_P T_0 = C_P T_e + \frac{u_e^2}{2}$
- $u_e^2 = 2C_P(T_0 - T_e)$
- $u_e = \left\{\frac{2\gamma R T_0}{\gamma-1}\left(1 - \frac{T_e}{T_0}\right)\right\}^{1/2}$ but $\left(\frac{P_o}{p}\right)^{\gamma-1/\gamma} = \left(\frac{T_o}{T}\right)$

Factors affecting exit velocity u_e

- So $\left(\frac{P_e}{P_0}\right)^{\gamma-1/\gamma} = \left(\frac{T_e}{T_0}\right)$
- $u_e = \left\{\frac{2\gamma}{\gamma-1}\frac{R_0}{MW}T_0\left(1-\left(\frac{P_e}{P_0}\right)^{\gamma-1/\gamma}\right)\right\}^{1/2}$
- Note: u_e increases with the following: see fig 2.7
  - Increased pressure ratio $\left({P_0}/{P_e}\right)$, although this is accompanied by an increase in motor weight
  - Increased combustion temperature, accompanied by the adverse effects of higher temperatures on nozzle heat transfer and dissociation of the reactants
  - Low molecular weight
  - Low γ, although this is of limited practicality

Gas velocity as a function of pressure ratio



gas velocity as a function of pressure ratio.

Exit to Throat Area Ratio

- $u_e (or\ V_e) = c^* C_F^0$ where $C_F^0 =$ characteristic thrust coefficient
- $C_F^0 = \left\{ \left[\gamma \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}} \right] \frac{2\gamma}{\gamma-1} \left(1 - \left(\frac{P_e}{P_0} \right)^{\gamma-1/\gamma} \right) \right\}^{1/2}$ this tells us about the performance of the nozzle
- And using continuity, $\dot{m} = \rho_x u_x A_x = constant$ it is possible to determine the exit to throat area ratio
- $\rho_e u_e A_e = \rho^* u^* A^*$
- $\frac{A_e}{A^*} = \frac{\rho^* u^*}{\rho_e u_e}$
- $\frac{A_e}{A^*} = \frac{\gamma \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}} \left(\frac{P_0}{P_e} \right)^{1/\gamma}}{C_F^0}$

Local velocity

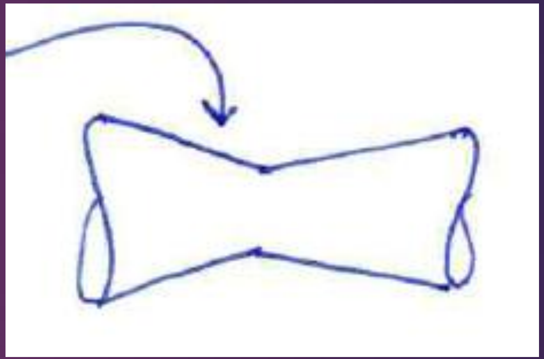
- For any nozzle $\dot{m} = \rho_x u_x A_x = constant$
- Therefore $\frac{\dot{m}}{A} = u\rho$
- The local velocity, u, can be obtained from the expression for the exit velocity
- $u \;\; = \left\{\frac{2\gamma}{\gamma - 1}\frac{R_0}{MW}T_0\left(1 - \left(\frac{P}{P_0}\right)^{\gamma-1/\gamma}\right)\right\}^{1/2}$
- Also $\quad \rho_0 = \frac{P_0}{RT_0} \quad ; \; \frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{1/\gamma}$
- So $\quad \rho = \rho_0\left(\frac{P}{P_0}\right)^{1/\gamma} = \frac{P_0}{RT_0}\left(\frac{P}{P_0}\right)^{1/\gamma}$

Design guide

- Hence the mass flow rate per unit area is given by
- $\frac{\dot{m}}{A} = \left\{\frac{2\gamma}{\gamma-1} RT_0 \left(1 - \left(\frac{P}{P_0}\right)^{\gamma-1/\gamma}\right)\right\}^{1/2} \frac{P_0}{RT_0} \left(\frac{P}{P_0}\right)^{1/\gamma}$
- $\frac{\dot{m}}{A} = P_0 \left\{\frac{2\gamma}{\gamma-1} \frac{1}{RT_0} \left(\frac{P}{P_0}\right)^{2/\gamma} \left(1 - \left(\frac{P}{P_0}\right)^{\gamma-1/\gamma}\right)\right\}^{1/2}$
- In terms of A
- $A = \frac{\dot{m}}{P_0} \left\{\frac{2\gamma}{\gamma-1} \frac{1}{RT_0} \left(\frac{P}{P_0}\right)^{2/\gamma} \left(1 - \left(\frac{P}{P_0}\right)^{\gamma-1/\gamma}\right)\right\}^{-1/2}$
- Which may be used to design a nozzle. Although the axial dimension is not represented. There is a degree of freedom.

Design guide

- A single nozzle could consist of two truncated cones joined at their narrowest section, which then becomes the throat.



- This would produce an appropriate expansion with the pressure and velocity adjusting in the cross-sectional area.
- There would however be considerable inefficiencies as the flow in the hypersonic gas stream would interact with the sharp edges at the joint generating shock waves.

Thrust equation

- We are now in a position to substitute values of velocity and mass flow rate into the thrust equation.
- $F = M\frac{dV}{dt} = \dot{m}V_e + A_e(P_e - P_a)$
- $F = P_0 A^* \left\{\frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(1-\left(\frac{P_e}{P_0}\right)^{\gamma-1/\gamma}\right)\right\}^{1/2} + A_e(P_e - P_a)$
- It is interesting to note that the molecular weight (MW) and combustion chamber temperature T_0 do not appear.
- They have been subsumed into the combustion chamber and exhaust pressures.

Design guide

- F = $M\frac{dV}{dt} = \dot{m}V_e + A_e(P_e - P_a)$
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- It is interesting to note that the molecular weight (MW) and combustion chamber temperature T_0 do not appear.
- They have been subsumed into the combustion chamber and exhaust pressures.
- $a \propto \rho \propto \left(\frac{MW}{T}\right)^{1/2}$ and
- $u \propto \left(\frac{T}{MW}\right)^{1/2}$

Design guide

- The main contribution to thrust comes from the mass flow rate – mostly determined by the throat area and combustion chamber pressure.
- The product $P_0 A^*$ is the fixed parameter which determines the size and general mechanical design of the rocket engine
- $A^* \Longrightarrow$ fixes overall dimensions
- $P_0 \Longrightarrow$ Determines the strength of the walls, pump capacity and dimensions
- The ratio $\frac{A_e}{A^*}$ is very important – sometimes called the expansion ratio
- $\frac{A_e}{A^*} \sim 10$ for first stage motors for use in low atmosphere
- $\frac{A_e}{A^*} \sim 80$ for high altitude and space
- For maximum efficiency $P_e = P_a$ and the value of P_e is determined by the expansion ratio $\frac{A_e}{A^*}$
- 2005 part ii; 2011 part b

End